CIfAR QH Summer school - May 26-28, 2009

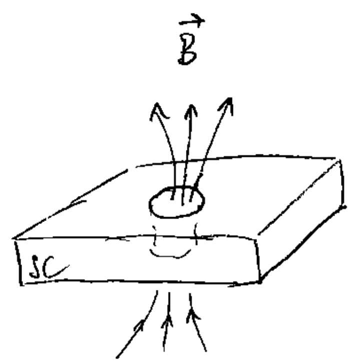
INTRO TO TOPOLOGICAL INSULATORS

Lecture I. "Exact quantization in solids"

- certain physical observables in solids are EXACTLY QUANTIZED, despite the fact that a solid can contain significant disorder:

1) Critical exponents
2) Magnetic flux in a SC
3) Hall conductance in IQHE & FQHE

(2)

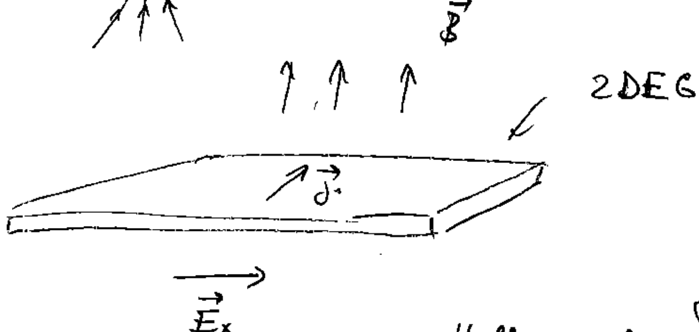


$$\Phi = \tfrac{1}{2} n \Phi_0 \qquad n \in \mathbb{Z}$$

$$\Phi_0 = \frac{hc}{e}$$

(flux quantum)

(3)



$$j_y = \sigma_{xy} E_x$$

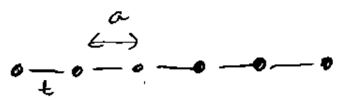
Hall conduct. $\sigma_{xy} = n \frac{e^2}{h}$ $\quad n \in \mathbb{Z}$ (IQHE)

<u>Why?</u> - SC and QHL possess <u>topological order</u> characterized by a <u>Topological invariant</u> that is insensitive to local perturbations.

• <u>Topological vs local order</u>

- <u>local order</u>: CDW, SDW, Peierls distortion ...

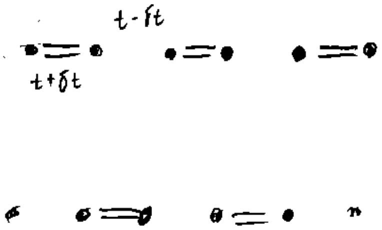
Example



↓

"dimerized state"

(1)



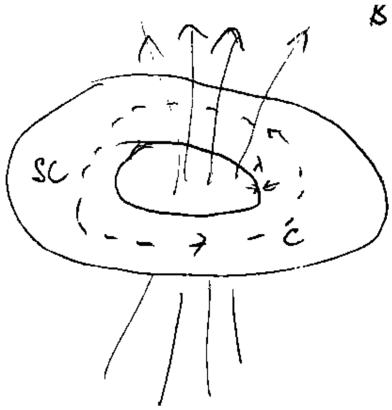
(2)

← broken symmetry

<u>has 2 degenerate ground states</u>

- <u>topological order</u> - ground state degeneracy that becomes apparent on the manifold with non-trivial topology and cannot be detected by local measurement.

Example: Flux quantization in SC



$$\oint_C \vec{j}\cdot d\vec{l} = 0$$

SC order parameter

$$\vec{j} = |\psi_0|^2 \frac{2e}{m}\left(\hbar\vec{\nabla}\theta - \frac{2e}{c}\vec{A}\right) \qquad \psi = |\psi_0| e^{i\theta}$$

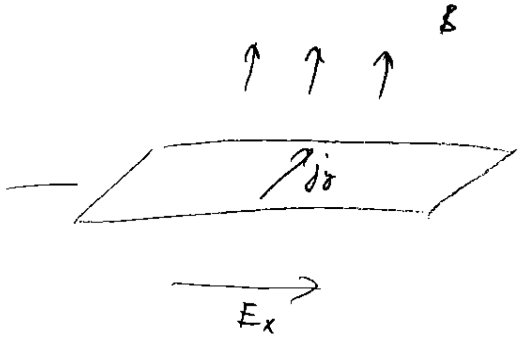
2e - Cooper pair charge

$$\oint_C \frac{\hbar c}{2e}\vec{\nabla}\theta\cdot d\vec{l} = \underbrace{\oint_C \vec{A}\cdot d\vec{l}}_{\Phi}$$

$$\Phi = \frac{\hbar c}{2e}\underbrace{\oint_c \vec{\nabla}\theta\cdot d\vec{l}}_{2\pi n} = \frac{1}{2} n \frac{hc}{2\pi e}\cdot 2\pi$$

$$= \frac{1}{2}\left(\frac{hc}{e}\right) n = \frac{1}{2}\Phi_0 n$$

• IQHE

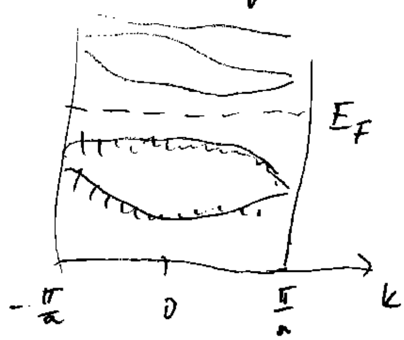


$$\sigma_{xy} = n \frac{e^2}{h}$$

$$n = \frac{i}{2\pi} \sum_{\substack{j \in occ \\ bands}} \int_{BZ} d^2k \left(\left\langle \frac{\partial \psi_i}{\partial k_x} \middle| \frac{\partial \psi_i}{\partial k_y} \right\rangle - \left\langle \frac{\partial \psi_i}{\partial k_y} \middle| \frac{\partial \psi_i}{\partial k_x} \right\rangle \right)$$

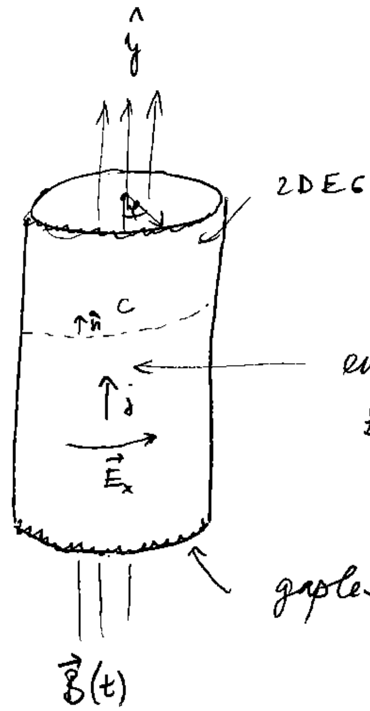
TKNN invariant (after Thouless - Kohmoto - Nightingale - den Nijs)

Assumption: Fermi level lies in the gap (insulator)



• Laughlin argument

- adiabatically ramp up the flux from 0 to Φ_0



energy gap
E_F in the gap.

gapless edge states

Faraday:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$j_y = \sigma_{xy} E_x$$

$$\frac{dQ}{dt} = -\oint_c dl\, \hat{n}\cdot\vec{j} = -\sigma_{xy}\oint_c d\vec{l}\cdot\vec{E}$$

$$= -\sigma_{xy}\int_s d\vec{s}\cdot(\vec{\nabla}\times\vec{E})$$

$$= \frac{\sigma_{xy}}{c}\int d\vec{s}\,\frac{d\vec{B}}{dt} = \frac{\sigma_{xy}}{c}\frac{d\Phi}{dt}$$

$$\boxed{\Delta Q = \frac{\sigma_{xy}}{c}\Delta\Phi}$$

Take $\Delta\Phi = \Phi_0$

$$\sigma_{xy} = \frac{c}{\Phi_0}\Delta Q = \frac{c}{\frac{hc}{e}}\Delta Q = \underline{\frac{e}{h}\Delta Q}$$

<u>What is ΔQ ?</u>

$$H_{2DEG}(\Phi=0) = H_{2DEG}(\Phi=\Phi_0)$$

↑ gauge invariance

$$H_{2DEG}^{(0)} = \frac{1}{2m}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2 + V(\vec{r})$$

$$H_{2DEC}(\phi_0) = \left(\vec{p} - \frac{e}{c}\vec{A} - \frac{e}{c}\delta\vec{A} \right)^2 + V(\vec{r})$$

$$\oint_C d\vec{l} \cdot \delta\vec{A} = \Phi_0 \qquad\qquad p = -i\hbar\vec{\nabla}$$

Gauge transf.
$$\vec{A} \to \vec{A} - \left(\frac{c\hbar}{e}\vec{\nabla}\lambda\right)$$
$$\Psi(\vec{r}) \to e^{i\lambda(\vec{r})}\Psi(\vec{r})$$

Take $\underline{\lambda(\vec{r}) = \varphi}$ identify $\delta\vec{A} = \frac{c\hbar}{e}\vec{\nabla}\lambda$

$$\oint d\vec{l} \cdot \delta\vec{A} = \frac{c\hbar}{e}\int d\vec{l} \cdot \vec{\nabla}\varphi = 2\pi\frac{\hbar c}{e} = \frac{hc}{e} = \Phi_0 \quad \checkmark$$

$\Rightarrow$ The eigenstate before and after flux insertion are identical.

- the many-body wavefunction can differ by occupancy of electron states.

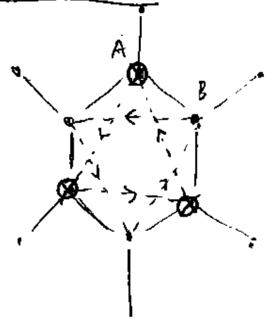
$$\Rightarrow \Delta Q = ne \qquad n \in \mathbb{Z}$$

$$\boxed{\sigma_{xy} = n\frac{e^2}{h}}$$

Lecture II : • IQHE without mag. field

Haldane model

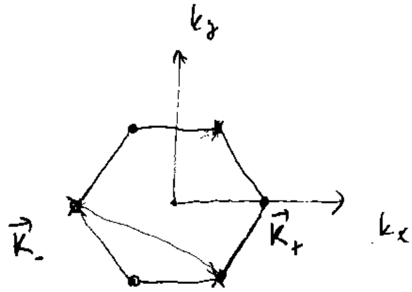
Graphene



$$H_0 = \sum_k (a_k^+, b_k^+)\, \mathcal{H}_k^{(0)} \begin{pmatrix} a_k \\ b_k \end{pmatrix}$$

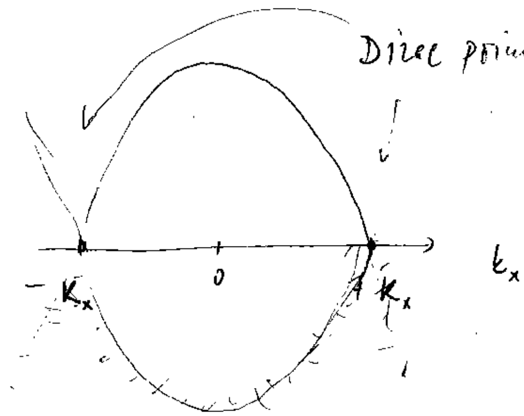
$$\mathcal{H}_k^{(0)} = \begin{pmatrix} 0 & t_k^* \\ t_k & 0 \end{pmatrix}$$

$$t_k = t \sum_{j=1}^{3} e^{i\vec{k}\cdot\vec{\delta}_j}$$



$$\vec{K}_\pm = \left(\pm \frac{2\pi}{3}, 0\right)$$

$$E_k = \pm |t_k|$$



Insulator: create a gap at $K_\pm$

$$\mathcal{H}_k^{(0)} = \sigma_1 \operatorname{Re} t_k + \sigma_2 \operatorname{Im} t_k = \sigma_1 \underbrace{\sum_j \cos \vec{k}\cdot\vec{\delta}_j}_{d_1(\vec{k})} + \sigma_2 \underbrace{\sum_j \sin \vec{k}\cdot\vec{\delta}_j}_{d_2(\vec{k})}$$

$$\mathcal{H}_k = \sigma_1 d_1(\vec{k}) + \sigma_2 d_2(\vec{k}) + \sigma_3 d_3(\vec{k}) \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}$$

Spectrum:

$$\mathcal{H}_k^2 = \mathbb{1}\left[d_1^2 + d_2^2 + d_3^2\right]$$

$$\Rightarrow \quad E_k^2 = d_1^2 + d_2^2 + d_3^2$$

$$\rightarrow \quad E_k = \pm\sqrt{d_1^2 + d_2^2 + d_3^2}$$

$$(\sigma_1 d_1)(\sigma_2 d_2) + (\sigma_2 d_2)(\sigma_1 d_1)$$

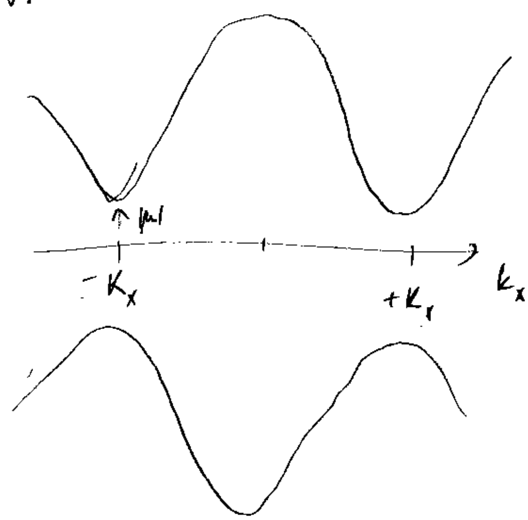
$$= d_1 d_2 \left(\sigma_1\sigma_2 + \sigma_2\sigma_1\right) = 0$$

Graphene: $d_3 = 0$ $\qquad E_k^{(\cdot)} = \pm\sqrt{d_1^2(\vec{k}) + d_2^2(\vec{k})}$

add $\underline{d_3(\vec{K}_\pm) \neq 0}$

(i) CDW (staggered potential) Semenoff (PRL 1984)

$$\delta\mathcal{H}_k^{CDW} = \begin{pmatrix} +\mu & 0 \\ 0 & -\mu \end{pmatrix} = \mu\,\sigma_3$$



$\longrightarrow$ opens up a gap $\Delta = 2|\mu|$

– "trivial insulator" $\qquad \sigma_{xy} = 0$

(ii) Haldane [PRL, 1988]

– second neighbor imaginary hopping $i\lambda$ (along the arrow)

$$\delta\mathcal{H}_k^{Hal} = \sigma_3\, 2\lambda \sum_j \sin(\vec{k}\cdot\vec{\gamma}_j) \qquad \vec{\gamma}_1 = \vec{\delta}_2 - \vec{\delta}_3, \ldots$$

gap $\Delta = 6\sqrt{3}|\lambda|$ at $K_\pm$

Time reversal $\mathcal{T}$: $\vec{k} \rightarrow -\vec{k}$

$\mathcal{H}_k^*$

- Calculation of TKNN invariant

$$\sigma_{xy} = \frac{e^2}{h} n$$

$$n = \frac{i}{2\pi} \int_{BZ} d^2k \left(\left\langle \frac{\partial \psi_-}{\partial k_x} \middle| \frac{\partial \psi_-}{\partial k_y} \right\rangle - \left\langle \frac{\partial \psi_-}{\partial k_y} \middle| \frac{\partial \psi_-}{\partial k_x} \right\rangle \right)$$

$\psi_-(\vec{k})$ - occupied band

$$\mathcal{H}_k \psi_-(\vec{k}) = E_k \psi_-(\vec{k})$$

- evaluation is CUMBERSOME

In the limit $\Delta \ll t$ principal contribution to the integral comes from vicinity of $\vec{K}_\pm$.

$\rightarrow$ linearize $\mathcal{H}_k$ near $\vec{K}_\pm$

$$\mathcal{H}^{(0)}_{k\pm} \simeq v(k_x \sigma_1 + k_y \sigma_2) \qquad v = \frac{3}{2} t \quad \text{Fermi velocity}$$

CDW: $\mathcal{H}^{CDW}_{k\pm} = \pm \mu \sigma_3$

Haldane $\mathcal{H}^{Hal}_{k\pm} = 3\sqrt{3}\lambda \sigma_3$

- TKNN invariant for a 2×2 Dirac Hamiltonian

$$h_k = k_x \sigma_1 + k_y \sigma_2 + m \sigma_3 \qquad (\tau = 1)$$

$$\psi_{k-} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\varphi_k \sqrt{1 - m/E_k} \\ \sqrt{1 + m/E_k} \end{pmatrix} \qquad \varphi_k = \frac{k_x - i k_y}{|k|}$$

$$E_k = \sqrt{k_x^2 + k_y^2 + m^2}$$

$$\frac{i}{2\pi} \int_{BZ} d^2k \left(\qquad \right) = \frac{1}{2} \operatorname{sgn}(m)$$

CDW: $\quad n = \frac{1}{2} \operatorname{sgn}(+\mu) + \frac{1}{2} \operatorname{sgn}(-\mu) = 0$

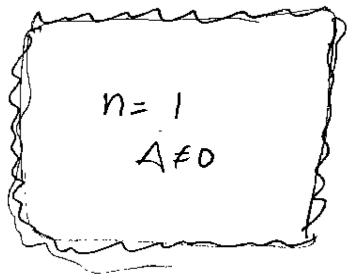
Hal: $\quad n = \frac{1}{2} \operatorname{sgn}(\lambda) + \frac{1}{2} \operatorname{sgn}(\lambda) = \operatorname{sgn}(\lambda) = \pm 1$

$$\boxed{\sigma_{xy} = \frac{e^2}{h} \operatorname{sgn}(\lambda)}$$

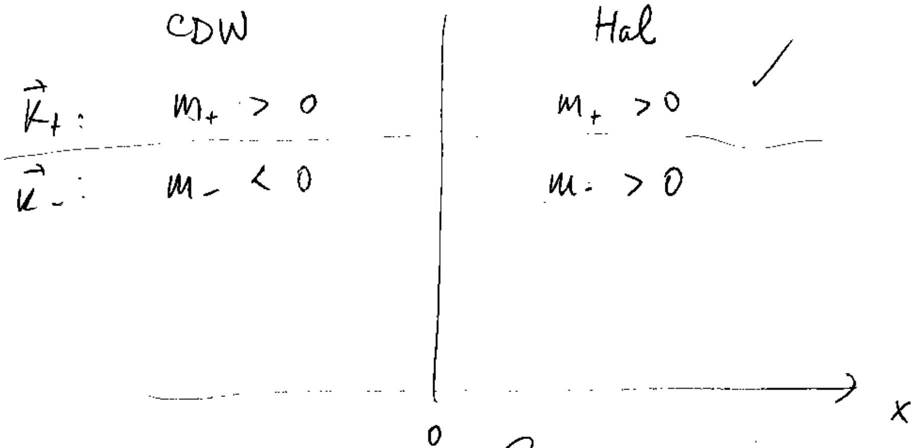
- exactly quantized, non-zero Hall conductance.

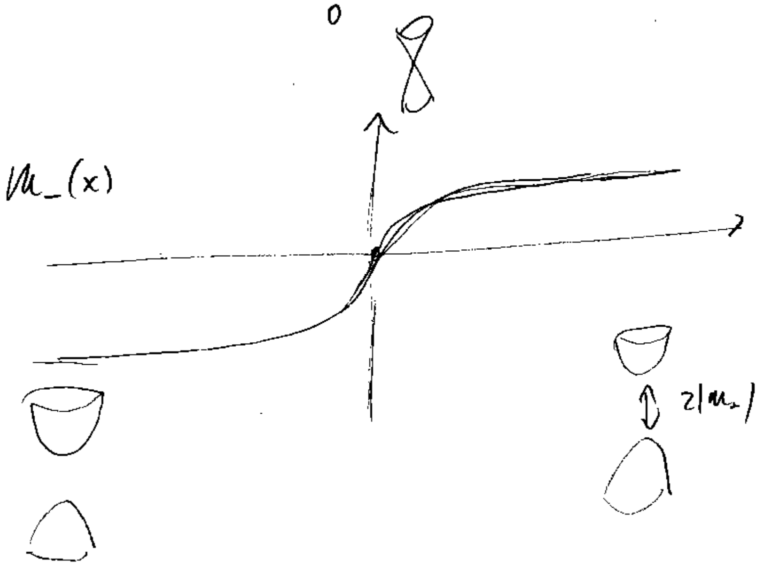
• Edge states

When $n \neq 0$ there necessarilly exist gaples edge states.



Consider a boundary





— soliton profile

$$h\psi = \varepsilon \psi$$

$$h = v(-i\partial_x \sigma_1 + k_y \sigma_2) + m(x)\sigma_3$$

$$\psi_{k_y}(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} e^{-\frac{1}{v}\int_0^x m(x')dx'}$$

$$\varepsilon_k = v k_y$$

↖ edge state

Topological insulator

- Haldane model for spinless electrons breaks T-invariance (2nn hopping imaginary, $\sigma_{xy} \neq 0$)

[**Q**: Can one have topological phase without breaking $\mathcal{T}$?]

A: Yes, if SO coupling is present.

Prototype: Kane-Mele model for graphene

$$H = H_0 + H_{SO}$$

$$H_{SO} = i\lambda \sum_{\substack{\langle\langle ij \rangle\rangle \\ \alpha,\beta}} (\vec{d}^{\,1}_{ij} \times \vec{d}^{\,2}_{ij}) \cdot \vec{\sigma}_{\alpha\beta}\, c^{\dagger}_{i\alpha} c_{j\beta}$$

$\downarrow$

Haldane term with $\lambda = +\lambda$ for spin $\uparrow$
$\lambda = -\lambda$ for spin $\downarrow$

- $\mathcal{T}$ invariant
- $\sigma_{xy} = 0$
- $\sigma^{s}_{xy} = \frac{e}{2\pi}$ $\qquad \vec{j}_s = \frac{\hbar}{2e}(\vec{j}_{\uparrow} - \vec{j}_{\downarrow})$